Pergamon

Int. J. Solids Structures Vol. 33, No. 6, pp. 865–878, 1996

0020-7683/96 $9.50 + .00

0020-7683(95)00062-3

THE CONSERVATION LAWS AND PATH-INDEPENDENT INTEGRALS WITH AN APPLICATION FOR LINEAR ELECTRO-MAGNETO-ELASTIC MEDIA

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(*Received* 20 *May* 1994; *in revised form* 18 *January* 1995)

Abstract— Based on the concept of the energy-momentum tensor, the paper derives the conservation laws relevant to free energy and complementary free energy of electro-magneto-elastic media. From those laws, the path-independent integrals are presented. Other laws and associated integrals are also introduced. Some integrals are evaluated to obtain the energy release rate for the mode III electro-magneto-elastic fracture problem.

1. INTRODUCTION

Since the J integral was introduced by Rice (1968) and widely applied in linear as well as non-linear fracture mechanics, the path-independent integral has received more and more attention. For the linear elastic media, the path-independent integrals can also be derived from conservation laws with the aid of Noether's theorem, as was done by Knowles and Sternberg (1972) and Fletcher (1976). Another method was introduced by Eshelby (1975) who deduced the path-independent integrals on the concept of the energy-momentum that first appeared in the classical electric field theory (Landau and Lifshitz, 1975). Corresponding to the path-independent integral, Bui (1974) set forward the dual path-independent integral associated with the elastic complementary energy, whereas Xu (1988) gave the dual conservation laws and more general dual path-independent integrals for elastostatics.

From Eshelby's method with the energy-momentum tensor, we can also obtain the conservation laws for electro-magneto-elastostatics provided that a state function, such as the strain energy and complementary energy in elastic theory, is independent of the position of the material point, or the material is homogeneous. As a consequence, the path-independent integrals and the associated energy release rate are obtained without any difficulty.

2. FUNDAMENTAL EQUATIONS

On account of the classical electromagnetic field and elastic theory, the fundamental equations for electro-magneto-elastic media are composed of

$$E_i = -\varphi_{,i} \tag{1a}$$

$$B_i = \mathbf{e}_{ijk}\mathbf{A}_{k,j} \tag{1b}$$

$$\boldsymbol{\varepsilon}_{ij} = (\mathbf{u}_{i,j}+\mathbf{u}_{j,i})/2 \quad \text{(small deformation)} \tag{1c}$$

$$D_{i,i} = 0 \tag{1d}$$

$$\mathbf{e}_{ijk}H_{k,j} = 0 \tag{1e}$$

$$\boldsymbol{\sigma}_{ij,i} = 0. \tag{1f}$$

where φ, E_i and D_i are the electric potential, electric field and electric displacement, respectively, $\mathbf{A}_i$, B_i and H_i are denoted as magnetic potential vector, magnetic induction and field, $\mathbf{u}_i$, $\boldsymbol{\varepsilon}_{ij}$ and $\boldsymbol{\sigma}_{ij}$ represent the displacement, strain and stress tensors and $\mathbf{e}_{ijk}$ is a permutation tensor. An alternative combination of eqn (1) is

$$
\begin{aligned}
E_i &= -\varphi_{,i} \\
H_i &= -\psi_{,i} \\
\boldsymbol{\varepsilon}_{ij} &= (\mathbf{u}_{i,j}+\mathbf{u}_{j,i})/2 \quad \text{(small deformation)} \\
D_{i,i} &= 0 \\
B_{i,i} &= 0 \\
\boldsymbol{\sigma}_{ij,j} &= 0,
\end{aligned}
\tag{2}
$$

where ψ is the magnetic potential variable.

We define the state functions, free energy ρf, ρF and complementary free energy ρg, ρG for linear material, to be

$$
\begin{aligned}
\rho f(\boldsymbol{\varepsilon}_{ij}, E_i, B_i) &= \tfrac{1}{2}(\alpha^{ss}_{ijkl}\boldsymbol{\varepsilon}_{ij}\boldsymbol{\varepsilon}_{kl} - \alpha^{ee}_{ij}E_iE_j - \alpha^{mm}_{ij}B_iB_j) - \alpha^{se}_{ijk}\varepsilon_{ij}E_k - \alpha^{sm}_{ijk}\varepsilon_{ij}B_k - \alpha^{em}_{ij}E_iB_j \\
\rho g(\boldsymbol{\sigma}_{ij}, D_i, H_i) &= \tfrac{1}{2}(-\xi^{ss}_{ijkl}\boldsymbol{\sigma}_{ij}\boldsymbol{\sigma}_{kl} + \xi^{ee}_{ij}D_iD_j + \xi^{mm}_{ij}H_iH_j) - \xi^{se}_{ijk}\boldsymbol{\sigma}_{ij}D_k - \xi^{sm}_{ijk}\boldsymbol{\sigma}_{ij}H_k + \xi^{em}_{ij}D_iH_j
\end{aligned}
\tag{3}
$$

and

$$
\begin{aligned}
\rho F(\boldsymbol{\varepsilon}_{ij}, E_i, H_i) &= \tfrac{1}{2}(\beta^{ss}_{ijkl}\varepsilon_{ij}\varepsilon_{kl} - \beta^{ee}_{ij}E_iE_j + \beta^{mm}_{ij}H_iH_j) - \beta^{se}_{ijk}\varepsilon_{ij}E_k + \beta^{sm}_{ijk}\varepsilon_{ij}H_k - \beta^{em}_{ij}E_iH_j \\
\rho G(\boldsymbol{\sigma}_{ij}, D_i, B_i) &= \tfrac{1}{2}(-\eta^{ss}_{ijkl}\boldsymbol{\sigma}_{ij}\boldsymbol{\sigma}_{kl} + \eta^{ee}_{ij}D_iD_j - \eta^{mm}_{ij}B_iB_j) - \eta^{se}_{ijk}\boldsymbol{\sigma}_{ij}D_k - \eta^{sm}_{ijk}\boldsymbol{\sigma}_{ij}B_k - \eta^{em}_{ij}D_iB_j,
\end{aligned}
\tag{4}
$$

where coefficients $\alpha^{(\ )}_{(\ )}$, $\beta^{(\ )}_{(\ )}$, $\xi^{(\ )}_{(\ )}$ and $\eta^{(\ )}_{(\ )}$ denote the characteristic of materials. Using the electro-magneto-thermodynamic theory presented by Wang (1993), we can obtain the constitutive relations as follows:

$$
\boldsymbol{\sigma}_{ij} = \frac{\partial \rho f}{\partial \boldsymbol{\varepsilon}_{ij}}, \quad D_i = -\frac{\partial \rho f}{\partial E_i}, \quad H_i = -\frac{\partial \rho f}{\partial B_i} \tag{5a}
$$

or

$$
\boldsymbol{\varepsilon}_{ij} = -\frac{\partial \rho f}{\partial \boldsymbol{\sigma}_{ij}}, \quad E_i = \frac{\partial \rho f}{\partial D_i}, \quad B_i = \frac{\partial \rho f}{\partial H_i} \tag{5b}
$$

or

$$
\boldsymbol{\sigma}_{ij} = \frac{\partial \rho F}{\partial \boldsymbol{\varepsilon}_{ij}}, \quad D_i = -\frac{\partial \rho F}{\partial E_i}, \quad B_i = \frac{\partial \rho f}{\partial H_i} \tag{5c}
$$

or

$$
\boldsymbol{\varepsilon}_{ij} = -\frac{\partial \rho G}{\partial \boldsymbol{\sigma}_{ij}}, \quad E_i = \frac{\partial \rho G}{\partial D_i}, \quad H_i = -\frac{\partial \rho G}{\partial B_i}. \tag{5d}
$$

3. CONSERVATION LAWS RELEVANT TO FREE ENERGY

In the following, we assume that the media are all homogeneous electro-magneto-elastic material. From the first definition in eqn (3) and the fundamental equation [eqn (1a–c)], we have

$$(\rho f)_{,k} = \frac{\partial \rho f}{\partial \varepsilon_{ij}}\mathbf{u}_{i,jk} - \frac{\partial \rho f}{\partial E_i}\varphi_{,ik} + \frac{\partial \rho f}{\partial B_i}\mathbf{e}_{ijm}\mathbf{A}_{m,jk}. \tag{6}$$

Using the constitutive relation (5a) and the fundamental equation [eqn (1d–f)], we arrive at

$$(\rho f)_{,k} = (\boldsymbol{\sigma}_{ij}\mathbf{u}_{i,k} + D_j\varphi_{,k} + \mathbf{e}_{ijm}H_m\mathbf{A}_{i,k})_{,j}. \tag{7}$$

Then we obtain the differential form of the first conservation equation

$$(\rho f\delta_{kj} - \boldsymbol{\sigma}_{ij}\mathbf{u}_{i,k} - D_j\varphi_{,k} - \mathbf{e}_{ijm}H_m\mathbf{A}_{i,k})_{,j} = 0 \tag{8}$$

whose integral expression is

$$\oint(\rho f\delta_{kj} - \boldsymbol{\sigma}_{ij}\mathbf{u}_{i,k} - D_j\varphi_{,k} - \mathbf{e}_{ijm}H_m\mathbf{A}_{i,k})n_j\,\mathrm{d}s = 0. \tag{9}$$

It is worth noting that eqns (8) and (9) hold if only the constitutive relation (5a) is available. It implies that it is not necessary for ρf to meet equality (3a), so either eqn (8) or eqn (9) is suitable for the problems with non-linear constitutive relations provided that the state function exists.

To proceed further, we define the energy-momentum tensor as

$$\mathbf{p}_{1kj} = \rho f\delta_{kj} - \boldsymbol{\sigma}_{ij}\mathbf{u}_{i,k} - D_j\varphi_{,k} - \mathbf{e}_{ijm}H_m\mathbf{A}_{i,k}. \tag{10}$$

When electric field is considered, we have

$$\mathbf{p}_{1kj} = E_kD_j - \tfrac{1}{2}E_iD_i\delta_{kj},$$

which is just an energy-momentum tensor in the static field or the so-called Maxwell stress tensor.

From the definition (3a) and the fundamental equation [eqn (1a–c)], we obtain

$$(x_j\mathbf{p}_{1jk})_{,k} = 3\rho f - (\boldsymbol{\sigma}_{ij}\mathbf{u}_{i,j} + D_j\varphi_{,j} + \mathbf{e}_{ijm}H_m\mathbf{A}_{i,j}). \tag{11}$$

Considering eqn (1d–f) and the free energy of a linear material

$$\rho f = \tfrac{1}{2}(\boldsymbol{\sigma}_{ij}\mathbf{u}_{i,j} + D_j\varphi_{,j} + \mathbf{e}_{ijm}H_m\mathbf{A}_{i,j}) \tag{12}$$

we have the second conservation equation

$$[x_j\mathbf{p}_{1jk} - \tfrac{1}{2}(\boldsymbol{\sigma}_{ik}\mathbf{u}_i + D_k\varphi + \mathbf{e}_{ikm}H_m\mathbf{A}_i)]_{,k} = 0 \tag{13}$$

whose integral expression is

$$\oint[x_j\mathbf{p}_{1jk} - \tfrac{1}{2}(\boldsymbol{\sigma}_{ik}\mathbf{u}_i + D_k\varphi + \mathbf{e}_{ikm}H_m\mathbf{A}_i)]n_k\,\mathrm{d}s = 0$$

or

$$\oint [x_k \rho f - x_j(\boldsymbol{\sigma}_{ik}\mathbf{u}_{i,j} + D_k\varphi_{,j} + \mathbf{e}_{ikm}H_m\mathbf{A}_{i,j}) - \tfrac{1}{2}(\boldsymbol{\sigma}_{ik}\mathbf{u}_i + D_k\varphi + \mathbf{e}_{ikm}H_m\mathbf{A}_i)]n_k\,\mathrm{d}s = 0. \quad (14)$$

As a result of eqns (9) and (14), the path-independent integrals are

$$J_{1k} = \int (\rho f\delta_{kj} - \boldsymbol{\sigma}_{ij}\mathbf{u}_{i,k} - D_j\varphi_{,k} - \mathbf{e}_{ijm}H_m\mathbf{A}_{i,k})n_j\,\mathrm{d}s$$

$$M_1 = \int [x_k\rho f - x_j(\boldsymbol{\sigma}_{ik}\mathbf{u}_{i,j} + D_k\varphi_{,j} + \mathbf{e}_{ikm}H_m\mathbf{A}_{i,j}) - \tfrac{1}{2}(\boldsymbol{\sigma}_{ik}\mathbf{u}_i + D_k\varphi + \mathbf{e}_{ikm}H_m\mathbf{A}_i)]n_k\,\mathrm{d}s. \quad (15)$$

For a piezoelectric material, eqn (15) is turned into

$$J_{1k} = \int (\rho f\delta_{kj} - \boldsymbol{\sigma}_{ij}\mathbf{u}_{i,k} - D_j\varphi_{,k})n_j\,\mathrm{d}s$$

$$M_1 = \int [x_k\rho f - x_j(\boldsymbol{\sigma}_{ik}\mathbf{u}_{i,j} + D_k\varphi_{,j}) - \tfrac{1}{2}(\boldsymbol{\sigma}_{ik}\mathbf{u}_i + D_k\varphi)]n_k\,\mathrm{d}s, \quad (16)$$

where eqn (16b) is in agreement with that given by Pak (1990a,b). In addition, regardless of the electromagnetic field, the path-independent integrals for elasticity are obtained and are the same as those derived by Knowles and Sternberg (1972) and Fletcher (1976).

In elasticity, there is the third conservation equation (Knowles and Sternberg, 1972; Fletcher, 1976; Xu, 1988) for linear isotropic materials. However, discussion of the third conservation equation for an electro-magneto-elastic material is meaningless due to the fact that there is no coupling between electricity, magnetics and elasticity for an isotropic material as long as its microstructure possesses the same symmetry group for electric, magnetic and elastic behaviour.

As is known, the static electro-magneto-elastic problem can also be determined by the alternative fundamental group of eqn (1), i.e. eqn (2). When it is applied, together with eqn (4a), it follows that

$$(\rho F)_{,k} = \frac{\partial \rho F}{\partial \boldsymbol{\varepsilon}_{ij}}\mathbf{u}_{i,jk} - \frac{\partial \rho F}{\partial E_i}\varphi_{,ik} - \frac{\partial \rho F}{\partial H_i}\psi_{,ik}$$
$$= (\boldsymbol{\sigma}_{ij}\mathbf{u}_{i,k} + D_j\varphi_{,k} - B_j\psi_{,k})_{,j}; \quad (17)$$

then another expression of the first conservation equation is

$$(\rho F\delta_{kj} - \boldsymbol{\sigma}_{ij}\mathbf{u}_{i,k} - D_j\varphi_{,k} + B_j\psi_{,k})_{,j} = 0 \quad (18)$$

and the energy-momentum tensor can also be defined as

$$\mathbf{p}_{1kj} = \rho F\delta_{kj} - \boldsymbol{\sigma}_{ij}\mathbf{u}_{i,k} - D_j\varphi_{,k} + B_j\psi_{,k}. \quad (19)$$

Considering

$$\rho F = \tfrac{1}{2}(\boldsymbol{\sigma}_{ik}\mathbf{u}_{i,k} + D_k\varphi_{,k} - B_k\psi_{,k})$$

we obtain another expression of the second conservation equation:

$$[x_j\mathbf{p}_{1jk} - \tfrac{1}{2}(\boldsymbol{\sigma}_{ik}\mathbf{u}_i + D_k\varphi - B_k\psi)]_{,k} = 0. \quad (20)$$

Therefore, the path-independent integrals are

$$J'_{1k} = \int (\rho F \delta_{kj} - \boldsymbol{\sigma}_{ij}\mathbf{u}_{i,k} - D_j\varphi_{,k} + B_j\psi_{,k})n_j \, ds$$

$$M'_1 = \int [x_k\rho F - x_j(\boldsymbol{\sigma}_{ik}\mathbf{u}_{i,j} + D_k\varphi_{,j} - B_k\psi_{,j}) - \tfrac{1}{2}(\boldsymbol{\sigma}_{ik}\mathbf{u}_i + D_k\varphi - B_k\psi)]n_k \, ds. \tag{21}$$

4. CONSERVATION LAWS REFERRED TO COMPLEMENTARY FREE ENERGY

If the material is homogeneous and the state function ρg [eqn (3b)] is adopted, we have

$$(\rho g)_{,k} = \frac{\partial \rho g}{\partial \boldsymbol{\sigma}_{ij}}\boldsymbol{\sigma}_{ij,k} + \frac{\partial \rho g}{\partial D_i} D_{i,k} + \frac{\partial \rho g}{\partial H_i} H_{i,k}. \tag{22}$$

With constitutive relation (5b) and the fundamental eqn (1a–c), eqn (22) is rewritten as

$$(\rho g)_{,k} = -u_{i,j}\boldsymbol{\sigma}_{ij,k} - \varphi_{,j}D_{j,k} + \mathbf{e}_{ijm}\mathbf{A}_{m,j}H_{i,k}. \tag{23}$$

Combining with the fundamental eqn (1d–f), the differential form of the first conservation equation relevant to complementary free energy is

$$(\rho g\delta_{kj} + \mathbf{u}_i\boldsymbol{\sigma}_{ij,k} + \varphi D_{j,k} - \mathbf{e}_{ijm}\mathbf{A}_m H_{i,k})_{,j} = 0 \tag{24}$$

whose integral expression is

$$\oint (\rho g\delta_{kj} + \mathbf{u}_i\boldsymbol{\sigma}_{ij,k} + \varphi D_{j,k} - \mathbf{e}_{ijm}\mathbf{A}_m H_{i,k})n_j \, ds = 0. \tag{25}$$

Similarly to eqns (8) and (9), eqns (24) and (25) are still suitable for the media with a non-linear constitutive relation provided that the state function ρg exists.

Let us suppose the energy-momentum tensor is

$$\mathbf{p}_{2kj} = \rho g\delta_{kj} + \mathbf{u}_i\boldsymbol{\sigma}_{ij,k} + \varphi D_{j,k} - \mathbf{e}_{ijm}\mathbf{A}_m H_{i,k} \tag{26}$$

and considering the fundamental eqn (1), we can obtain

$$(x_j\mathbf{p}_{2jk})_{,k} = 3\rho g. \tag{27}$$

Since

$$\rho g = \tfrac{1}{2}(-\mathbf{u}_{i,k}\boldsymbol{\sigma}_{ik} - \varphi_{,k}D_k + \mathbf{e}_{ikm}\mathbf{A}_{m,k}H_i).$$

the second conservation equation is described by

$$[x_j\mathbf{p}_{2jk} + \tfrac{3}{2}(\mathbf{u}_i\boldsymbol{\sigma}_{ik} + \varphi D_k - \mathbf{e}_{ikm}\mathbf{A}_m H_i)]_{,k} = 0 \tag{28}$$

whose integral expression is

$$\oint [x_j\mathbf{p}_{2jk} + \tfrac{3}{2}(\mathbf{u}_i\boldsymbol{\sigma}_{ik} + \varphi D_k - \mathbf{e}_{ikm}\mathbf{A}_m H_i)]n_k \, ds = 0. \tag{29}$$

Corresponding to eqns (25) and (29), the path-independent integrals are

$$J_{2k} = \int (\rho g \delta_{kj} - \mathbf{u}_i \boldsymbol{\sigma}_{ij,k} + \varphi D_{j,k} - \mathbf{e}_{ijm} \mathbf{A}_{mi} H_{i,k}) n_j \, \mathrm{d}s$$

$$M_2 = \int [x_k \rho g + x_j (u_i \boldsymbol{\sigma}_{ik,j} + \varphi D_{k,j} - \mathbf{e}_{ikm} \mathbf{A}_m H_{i,j}) + \tfrac{3}{2}(u_i \boldsymbol{\sigma}_{ik} + \varphi D_k - \mathbf{e}_{ikm} \mathbf{A}_m H_i)] n_k \, \mathrm{d}s. \quad (30)$$

For a piezoelectric material, eqn (29) is simplified into

$$J_{2k} = \int (\rho g \delta_{kj} + \mathbf{u}_i \boldsymbol{\sigma}_{ij,k} + \varphi D_{j,k}) n_j \, \mathrm{d}s$$

$$M_2 = \int [x_k \rho g + x_j (\mathbf{u}_i \boldsymbol{\sigma}_{ik,j} + \varphi D_{k,j}) + \tfrac{3}{2}(\mathbf{u}_i \boldsymbol{\sigma}_{ik} + \varphi D_k)] n_k \, \mathrm{d}s. \quad (31)$$

Without regard to the electromagnetic field, eqn (31) is identical with that given by Xu (1988). If eqn (31a) degenerates further into a two-dimensional problem, Bui's (1974) result is obtained.

As the state function, complementary free energy ρG, is applied together with the fundamental eqn (2), we have

$$\begin{aligned}(\rho G)_{,k} &= \frac{\partial \rho G}{\partial \boldsymbol{\sigma}_{ij}} \boldsymbol{\sigma}_{ij,k} + \frac{\partial \rho G}{\partial D_j} D_{j,k} - \frac{\partial \rho G}{\partial B_j} \psi_{j,k} \\ &= -\mathbf{u}_{i,j} \boldsymbol{\sigma}_{ij,k} - \varphi_{,j} D_{j,k} + \psi_{,j} B_{j,k} \end{aligned} \quad (32)$$

and the first conservation equation relevant to a complementary free energy is

$$(\rho G \delta_{kj} + \mathbf{u}_i \boldsymbol{\sigma}_{ij,k} + \varphi_j D_{j,k} - \psi_j B_{j,k})_{,j} = 0. \quad (33)$$

With reference to eqn (33), the related energy-momentum tensor can be defined as

$$\mathbf{p}_{2kj} = \rho G \delta_{kj} + \mathbf{u}_i \boldsymbol{\sigma}_{ij,k} + \varphi D_{j,k} - \psi B_{j,k}; \quad (34)$$

moreover

$$(x_j \mathbf{p}_{2jk})_{,k} = 3\rho G. \quad (35)$$

Since

$$\rho G = \tfrac{1}{2}(-\mathbf{u}_{i,k} \boldsymbol{\sigma}_{ik} - \varphi_{,k} D_k + \psi_{,k} B_k) \quad (36)$$

the second conservation equation is introduced by

$$[x_j \mathbf{p}_{2jk} - \tfrac{3}{2}(\mathbf{u}_i \boldsymbol{\sigma}_{ik} + \varphi D_k - \psi B_k)]_{,k} = 0. \quad (37)$$

Consequent upon eqn (36), the path-independent integrals are

$$J'_{2k} = \int (\rho G \delta_{kj} + \mathbf{u}_i \boldsymbol{\sigma}_{ij,k} + \varphi D_{j,k} - \psi B_{j,k}) n_j \, \mathrm{d}s$$

$$M'_2 = \int [x_k \rho G + x_j (\mathbf{u}_i \boldsymbol{\sigma}_{ik,j} + \varphi D_{k,j} - \psi B_{k,j}) + \tfrac{3}{2}(\mathbf{u}_i \boldsymbol{\sigma}_{ik} + \varphi D_k - \psi B_k)] n_k \, \mathrm{d}s. \quad (38)$$

5. OTHER CONSERVATION EQUATIONS AND PATH—INDEPENDENT INTEGRALS

In Sections 3 and 4, the free energy and the complementary free energy described in Section 2 are taken as state functions. In fact, other state functions can also be applied to obtain the conservation laws and the path-independent integrals, which are to be presented without the procedure of derivation as follows.

Suppose that

$$\rho f_1(\boldsymbol{\varepsilon}_{ij}, D_i, H_i) = \tfrac{1}{2}(\gamma^{ss}_{ijkl}\boldsymbol{\varepsilon}_{ij}\boldsymbol{\varepsilon}_{kl} + \gamma^{ee}_{ij}D_iD_j + \gamma^{mm}_{ij}H_iH_j) + \gamma^{se}_{ijk}\boldsymbol{\varepsilon}_{ij}D_k + \gamma^{sm}_{ijk}\boldsymbol{\varepsilon}_{ij}H_k + \gamma^{em}_{ij}D_iH_j$$

$$\rho g_1(\boldsymbol{\sigma}_{ij}, E_i, B_i) = \tfrac{1}{2}(-\theta^{ss}_{ijkl}\boldsymbol{\sigma}_{ij}\boldsymbol{\sigma}_{kl} - \theta^{ee}_{ij}E_iE_j - \theta^{mm}_{ij}B_iB_j) - \theta^{se}_{ijk}\boldsymbol{\sigma}_{ij}E_k - \theta^{sm}_{ijk}\boldsymbol{\sigma}_{ij}B_k - \theta^{em}_{ij}E_iB_j, \quad (39)$$

where $\gamma^{\{\ \}}_{(\ )}$ and $\theta^{\{\ \}}_{(\ )}$ describe the character of materials. For convenience, the coefficients introduced like $\gamma^{\{\ \}}_{(\ )}$ and $\theta^{\{\ \}}_{(\ )}$ are not explained again in the following. It is noted that the state function ρg_1 is indeed the internal energy of electro-magneto-elastic media.

From eqns (1) and (39), we have

$$(\rho f_1\delta_{kj} - \boldsymbol{\sigma}_{ij}\mathbf{u}_{i,k} + \varphi D_{j,k} - \mathbf{e}_{ijm}\mathbf{A}_mH_{i,k})_{,j} = 0$$

$$[x_k\rho f_1 + x_j(-\boldsymbol{\sigma}_{ik}\mathbf{u}_{i,j} + \varphi D_{k,j} - \mathbf{e}_{ikm}\mathbf{A}_mH_{i,j}) + \tfrac{1}{2}(-\boldsymbol{\sigma}_{ik}\mathbf{u}_i + 3D_k\varphi + 3\mathbf{e}_{ikm}H_m\mathbf{A}_i)]_{,k} = 0$$

$$(\rho g_1\delta_{kj} + \mathbf{u}_i\boldsymbol{\sigma}_{ij,k} - D_j\varphi_{,k} - \mathbf{e}_{ijm}H_m\mathbf{A}_{i,k})_{,j} = 0$$

$$[x_k\rho g_1 + x_j(\mathbf{u}_i\boldsymbol{\sigma}_{ik,j} - D_k\varphi_{,j} - \mathbf{e}_{ikm}H_m\mathbf{A}_{i,j}) + \tfrac{1}{2}(3\mathbf{u}_i\boldsymbol{\sigma}_{ik} + 3\varphi D_k - \mathbf{e}_{ikm}H_m\mathbf{A}_i)]_{,k} = 0 \quad (40)$$

and the corresponding path-independent integrals are

$$J_{3k} = \int (\rho f_1\delta_{kj} - \boldsymbol{\sigma}_{ij}\mathbf{u}_{i,k} + \varphi D_{j,k} - \mathbf{e}_{ijm}\mathbf{A}_mH_{i,k})n_j\,\mathrm{d}s$$

$$M_3 = \int [x_k\rho f_1 + x_j(-\boldsymbol{\sigma}_{ik}\mathbf{u}_{i,j} + \varphi D_{k,j} - \mathbf{e}_{ikm}\mathbf{A}_mH_{i,j}) + \tfrac{1}{2}(-\boldsymbol{\sigma}_{ik}\mathbf{u}_i + 3D_k\varphi + 3\mathbf{e}_{ikm}H_m\mathbf{A}_i)]n_k\,\mathrm{d}s$$

$$J_{4k} = \int (\rho g_1\delta_{kj} + \mathbf{u}_i\boldsymbol{\sigma}_{ij,k} - D_j\varphi_{,k} - \mathbf{e}_{ijm}H_m\mathbf{A}_{i,k})n_j\,\mathrm{d}s$$

$$M_4 = \int [x_k\rho g_1 + x_j(\mathbf{u}_i\boldsymbol{\sigma}_{ik,j} - D_k\varphi_{,j} - \mathbf{e}_{ikm}H_m\mathbf{A}_{i,j}) + \tfrac{1}{2}(3\mathbf{u}_i\boldsymbol{\sigma}_{ik} + 3\varphi D_k - \mathbf{e}_{ikm}H_m\mathbf{A}_i)]n_k\,\mathrm{d}s. \quad (41)$$

Supposing that

$$\rho f_2(\boldsymbol{\sigma}_{ij}, E_i, H_i) = \tfrac{1}{2}(-\kappa^{ss}_{ijkl}\boldsymbol{\sigma}_{ij}\boldsymbol{\sigma}_{kl} - \kappa^{ee}_{ij}E_iE_j + \kappa^{mm}_{ij}H_iH_j) - \kappa^{se}_{ijk}\boldsymbol{\sigma}_{ij}E_k - \kappa^{sm}_{ijk}\boldsymbol{\sigma}_{ij}H_k - \kappa^{em}_{ij}E_iH_j$$

$$\rho g_2(\boldsymbol{\varepsilon}_{ij}, D_i, B_i) = \tfrac{1}{2}(v^{ss}_{ijkl}\boldsymbol{\varepsilon}_{ij}\boldsymbol{\varepsilon}_{kl} + v^{ee}_{ij}D_iD_j - v^{mm}_{ij}B_iB_j) + v^{se}_{ijk}\boldsymbol{\varepsilon}_{ij}D_k - v^{sm}_{ijk}\boldsymbol{\varepsilon}_{ij}B_k - v^{em}_{ij}D_iB_j. \quad (42)$$

From eqns (1) and (42), the conservation laws are described as

$$(\rho f_2\delta_{kj} + \mathbf{u}_i\boldsymbol{\sigma}_{ij,k} - D_j\varphi_{,k} - \mathbf{e}_{ijm}A_mH_{i,k})_{,j} = 0$$

$$[x_k\rho f_2 + x_j(\mathbf{u}_i\boldsymbol{\sigma}_{ik,j} - D_k\varphi_{,j} - \mathbf{e}_{ikm}\mathbf{A}_mH_{i,j}) + \tfrac{1}{2}(3\mathbf{u}_i\boldsymbol{\sigma}_{ik} - D_k\varphi + 3\mathbf{e}_{ikm}H_m\mathbf{A}_i)]_{,k} = 0$$

$$(\rho g_2\delta_{kj} - \boldsymbol{\sigma}_{ij}\mathbf{u}_{i,k} + \varphi D_{j,k} - \mathbf{e}_{ijm}H_m\mathbf{A}_{i,k})_{,j} = 0$$

$$[x_k\rho g_2 + x_j(-\boldsymbol{\sigma}_{ik}\mathbf{u}_{i,j} + \varphi D_{k,j} - \mathbf{e}_{ikm}H_m\mathbf{A}_{i,j}) + \tfrac{1}{2}(-\boldsymbol{\sigma}_{ik}\mathbf{u}_i + 3\varphi D_k - \mathbf{e}_{ikm}H_m\mathbf{A}_i)]_{,k} = 0 \quad (43)$$

and the corresponding path-independent integrals are

$$J_{5k} = \int (\rho f_2 \delta_{kj} + \mathbf{u}_i \boldsymbol{\sigma}_{ij,k} - D_j \varphi_{,k} - \mathbf{e}_{ijm} \mathbf{A}_m H_{i,k}) n_j \, ds$$

$$M_5 = \int [x_k \rho f_2 + x_j (\mathbf{u}_i \boldsymbol{\sigma}_{ik,j} - D_k \varphi_{,j} - \mathbf{e}_{ikm} \mathbf{A}_m H_{i,j}) + \tfrac{1}{2}(3\mathbf{u}_i \boldsymbol{\sigma}_{ik} - D_k \varphi + 3\mathbf{e}_{ikm} H_m \mathbf{A}_i)] n_k \, ds$$

$$J_{6k} = \int (\rho g_2 \delta_{kj} - \boldsymbol{\sigma}_{ij} \mathbf{u}_{i,k} + \varphi D_{j,k} - \mathbf{e}_{ijm} H_m \mathbf{A}_{i,k}) n_j \, ds$$

$$M_6 = \int [x_k \rho g_2 + x_j (-\boldsymbol{\sigma}_{ik} \mathbf{u}_{i,j} + \varphi D_{k,j} - \mathbf{e}_{ikm} H_m \mathbf{A}_{i,j}) + \tfrac{1}{2}(-\boldsymbol{\sigma}_{ik} \mathbf{u}_i + 3\varphi D_k - \mathbf{e}_{ikm} H_m \mathbf{A}_i)] n_k \, ds. \quad (44)$$

If eqn (1) is applied and eqn (4) is considered as a state function, we obtain

$$(\rho F \delta_{kj} - \boldsymbol{\sigma}_{ij} \mathbf{u}_{i,k} - D_j \varphi_{,k} - \mathbf{e}_{ijm} \mathbf{A}_m H_{i,k})_{,j} = 0$$
$$[x_k \rho F + x_j(-\boldsymbol{\sigma}_{ik} \mathbf{u}_{i,j} - D_k \varphi_{,j} - \mathbf{e}_{ikm} \mathbf{A}_m H_{i,j}) + \tfrac{1}{2}(-\boldsymbol{\sigma}_{ik} \mathbf{u}_i - D_k \varphi + 3\mathbf{e}_{ikm} H_m \mathbf{A}_i)]_{,k} = 0$$
$$(\rho G \delta_{kj} + \mathbf{u}_i \boldsymbol{\sigma}_{ij,k} + \varphi D_{j,k} - \mathbf{e}_{ijm} H_m \mathbf{A}_{i,k})_{,j} = 0$$
$$[x_k \rho G + x_j(\mathbf{u}_i \boldsymbol{\sigma}_{ik,j} + \varphi D_{k,j} - \mathbf{e}_{ikm} H_m \mathbf{A}_{i,j}) + \tfrac{1}{2}(3\mathbf{u}_i \boldsymbol{\sigma}_{ik} + 3\varphi D_k - \mathbf{e}_{ikm} H_m \mathbf{A}_i)]_{,k} = 0 \quad (45)$$

and the corresponding path-independent integrals are

$$J_{7k} = \int (\rho F \delta_{kj} - \boldsymbol{\sigma}_{ij} \mathbf{u}_{i,k} - D_j \varphi_{,k} - \mathbf{e}_{ijm} A_m H_{i,k}) n_j \, ds$$

$$M_7 = \int [x_k \rho F + x_j(-\boldsymbol{\sigma}_{ik} \mathbf{u}_{i,j} - D_k \varphi_{,j} - \mathbf{e}_{ikm} \mathbf{A}_m H_{i,j}) + \tfrac{1}{2}(-\boldsymbol{\sigma}_{ik} \mathbf{u}_i - D_k \varphi + 3\mathbf{e}_{ikm} H_m \mathbf{A}_i)] n_k \, ds$$

$$J_{8k} = \int (\rho G \delta_{kj} + \mathbf{u}_i \boldsymbol{\sigma}_{ij,k} + \varphi D_{j,k} - \mathbf{e}_{ijm} H_m \mathbf{A}_{i,k}) n_j \, ds$$

$$M_8 = \int [x_k \rho G + x_j(\mathbf{u}_i \boldsymbol{\sigma}_{ik,j} + \varphi D_{k,j} - \mathbf{e}_{ikm} H_m \mathbf{A}_{i,j}) + \tfrac{1}{2}(3\mathbf{u}_i \boldsymbol{\sigma}_{ik} + 3\varphi D_k - \mathbf{e}_{ikm} H_m \mathbf{A}_i)] n_k \, ds. \quad (46)$$

Certainly, if combining the above state function, and the fundamental eqn (2) is applied, we can further achieve other conservation laws and the path-independent integrals derived as in Sections 3 and 4.

6. THE ENERGY RELEASE RATE FOR MODE III FRACTURE

As shown in Fig. 1, for a crack in an infinite body subjected to out-of-plane deformation, two kinds of boundary conditions are introduced on its face:

(I) $\sigma_{zy} = 0$ (traction-free)
$D_y = 0$ (of no residence charge and negligible D_y in crack)
$H_x = 0$ (of no residence electric current and negligible H_x in crack); (47)

(II) $\sigma_{zy} = 0$ (traction-free)
$D_y = 0$ (of no residence charge and negligible D_y in crack)
$B_y = 0$ (negligible B_y in crack). (48)

If the crack tip field is known, the energy release rate can be obtained by using the first kind of path-independent integral, such as eqns (15a), (21a), (30a), etc.

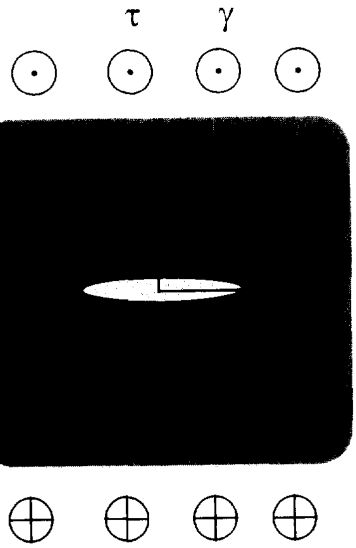

Fig. 1. The infinite body with a crack suffering from out-of-plane deformation.

Supposing

$$\begin{aligned} &u_x = u_y = 0, \quad u_z = u_z(x,y) \\ &\varphi = \varphi(x,y) \\ &A_x = A_y = 0, \quad A_z = A_z(x,y) \end{aligned} \tag{49}$$

and considering the boundary condition (47), we can obtain the electro-magneto-elastic mode III crack tip field (B2) derived in Appendix B. With the substitution of eqn (B2) into the path-independent integral (15a), the energy release rate is

$$J_{11} = \tfrac{1}{2}(c_{44}K^S K^S - \varepsilon_{11}K^E K^E + \mu_{11}K^B K^B) + e_{15}K^S K^E. \tag{50}$$

If the boundary condition satisfies eqn (48), the crack tip field could be presented by eqn (B4) (see Appendix B). Then the energy release rate is

$$J_{11} = \tfrac{1}{2}(c_{44}K^S K^S - \varepsilon_{11}K^E K^E) + e_{15}K^S K^E. \tag{51}$$

If we add another supposition to the field, that is

$$\begin{aligned} &u_x = u_y = 0, \quad u_z = u_z(x,y) \\ &\varphi = \varphi(x,y) \\ &\psi = \psi(x,y) \end{aligned} \tag{52}$$

we can obtain the crack tip field (B6) and (B8) corresponding to the boundary condition (47) and (48), respectively (see Appendix B). Referring to eqn (21a), the energy release rate is

$$J'_{11} = \tfrac{1}{2}(c^*_{44}K^S K^S - \varepsilon^*_{11}K^E K^E) + e^*_{15}K^S K^E \tag{53}$$

and

$$J'_{II} = \tfrac{1}{2}(c^*_{44}K^S K^S - \epsilon^*_{11}K^E K^E + \mu^*_{11}K^H K^H) + e^*_{15}K^S K^E - f^*_{15}K^S K^H - g^*_{11}K^H K^E. \quad (54)$$

It is found that the energy release rates deduced from eqn (30a), (41a,c), (44a,c) and (46a,c) are all identical to eqns (50) and (51) as the crack tip fields (B2) and (B4) are considered, whereas one derived from eqn (38a) is the same as eqn (53) or (54) as (B6) or (B8) is used. Furthermore, eqns (51) and (53) show that the magnetic field makes no contribution towards the energy release rate provided that the boundary condition (48) and the supposition (49), or (47) and (52) are adopted. Moreover, eqns (50), (51), (53) and (54) imply that the electric field has a tendency to retard the growth of a mode III crack, but the magnetic field has a contrary effect. It is also noted that no coupling appears between the magnetic field and elastic or electric field in eqn (50) upon consideration of the boundary condition (47) and the assumption (49).

7. CONCLUSION

In light of the view of Eshelby's energy-momentum tensor, the paper has presented many kinds of conservation laws and path-independent integrals. Even though all state functions are introduced to the quadratic form in the paper, the first kind of path-independent integral, such as eqns (15a), (21a), (30a), (38a), etc. are still appropriate for the non-linear electro-magneto-elastic media. It implies that the polynomial power of the state function may reach more than two.

In addition, it is shown that the energy release rate for fracture can be easily obtained by using the path-independent integral. The results further show that the the electric field has an inclination to alleviate the energy release rate of an electro-magneto-elastic mode III crack, but the magnetic field plays the inverse role on the crack.

Acknowledgement This work was supported by the foundation of National Education Committee.

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APPENDIX A: THE SOLUTION OF AN OUT-OF-DEFORMED BODY

In the following, both fundamental equations (1) and (2) are applied in the solution of the out-of-plane deformation problem for a transversely isotropic elasto-magneto-elastic medium. The normal of the plane coincides with the Z-axis.

Solution I

According to the constitutive relation (5a), the relation for the transversely isotropic media can be simplified to

$$\begin{aligned} \{\boldsymbol{\sigma}\} &= [\mathbf{C}]\{\varepsilon\} - [\mathbf{e}]^T\{\mathbf{E}\} - [\mathbf{f}]^T\{\mathbf{B}\} \\ \{\mathbf{D}\} &= [\mathbf{e}]\{\varepsilon\} - [\boldsymbol{\epsilon}]\{\mathbf{E}\} + [\mathbf{g}]\{\mathbf{B}\} \\ \{\mathbf{H}\} &= [\mathbf{f}]\{\varepsilon\} - [\mathbf{g}]\{\mathbf{E}\} + [\mathbf{u}]\{\mathbf{B}\}, \end{aligned} \quad (A1)$$

where

$$\{\boldsymbol{\sigma}\}^{\mathrm{T}} = \{\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{zy}, \sigma_{zx}, \sigma_{xy}\}$$

$$\{\boldsymbol{\varepsilon}\}^{\mathrm{T}} = \{\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}, \gamma_{zy}, \gamma_{zx}, \gamma_{xy}\}$$

$$\{\mathbf{E}\}^{\mathrm{T}} = \{E_x, E_y, E_z\}, \quad \{\mathbf{D}\}^{\mathrm{T}} = \{D_x, D_y, D_z\}$$

$$\{\mathbf{B}\}^{\mathrm{T}} = \{B_x, B_y, B_z\}, \quad \{\mathbf{H}\}^{\mathrm{T}} = \{H_x, H_y, H_z\}$$

$$[\mathbf{C}] = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{11} & c_{13} & 0 & 0 & 0 \\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & (c_{11}-c_{12})/2 \end{bmatrix}$$

$$[\mathbf{e}] = \begin{bmatrix} 0 & 0 & 0 & 0 & e_{15} & 0 \\ 0 & 0 & 0 & e_{15} & 0 & 0 \\ e_{31} & e_{31} & e_{33} & 0 & 0 & 0 \end{bmatrix}, \quad [\mathbf{f}] = \begin{bmatrix} 0 & 0 & 0 & 0 & f_{15} & 0 \\ 0 & 0 & 0 & f_{15} & 0 & 0 \\ f_{31} & f_{31} & f_{33} & 0 & 0 & 0 \end{bmatrix}$$

$$[\boldsymbol{\epsilon}] = \begin{bmatrix} \epsilon_{11} & 0 & 0 \\ 0 & \epsilon_{11} & 0 \\ 0 & 0 & \epsilon_{33} \end{bmatrix}, \quad [\mathbf{g}] = \begin{bmatrix} g_{11} & 0 & 0 \\ 0 & g_{11} & 0 \\ 0 & 0 & g_{33} \end{bmatrix}$$

$$[\boldsymbol{\mu}] = \begin{bmatrix} \mu_{11} & 0 & 0 \\ 0 & \mu_{11} & 0 \\ 0 & 0 & \mu_{33} \end{bmatrix}.$$

Supposing

$$u_x = u_y = 0, \quad u_z = u_z(x,y) \quad \text{(out-of-plane deformation)}$$
$$A_x = A_y = 0, \quad A_z = A_z(x,y), \quad \varphi = \varphi(x,y) \tag{A2}$$

and substituting eqn (A1) into the fundamental eqn (1d–f), we obtain

$$c_{44}\nabla^2 u_z + e_{15}\nabla^2\varphi = 0$$
$$e_{15}\nabla^2 u_z - \epsilon_{11}\nabla^2\varphi = 0$$
$$\mu_{11}\nabla^2 A_z = 0. \tag{A3}$$

Due to the arbitary choice of coefficients, eqn (A2) leads to

$$\nabla^2 u_z = 0, \quad \nabla^2\varphi = 0, \quad \nabla^2 A_z = 0. \tag{A4}$$

From eqn (A4), we get

$$u_z = \mathrm{Im}\,[U_1(z)], \quad \varphi = \mathrm{Im}\,[\Phi_1(z)], \quad A_z = \mathrm{Im}\,[\Omega(z)], \tag{A5}$$

where $U_1(z)$, $\Phi_1(z)$ and $\Omega(z)$ are complex potentials, $z = x+iy$. As a consequence, the strain, stress, electric field, electric displacement, magnetic induction and magnetic field are described by

$$\begin{cases}\gamma_{zx} = \mathrm{Im}\,[U_1'(z)], & \sigma_{zx} = \mathrm{Im}\,[c_{44}U_1'(z)+e_{15}\Phi_1'(z)]+\mathrm{Re}\,[f_{15}\Omega'(z)] \\ \gamma_{zy} = \mathrm{Re}\,[U_1'(z)], & \sigma_{zy} = \mathrm{Re}\,[c_{44}U_1'(z)+e_{15}\Phi_1'(z)]-\mathrm{Im}\,[f_{15}\Omega'(z)]\end{cases}$$

$$\begin{cases}E_x = -\mathrm{Im}\,[\Phi_1'(z)], & D_x = \mathrm{Im}\,[e_{15}U_1'(z)-\epsilon_{11}\Phi_1'(z)]-\mathrm{Re}\,[g_{11}\Omega'(z)] \\ E_y = -\mathrm{Re}\,[\Phi_1'(z)], & D_y = \mathrm{Re}\,[e_{15}U_1'(z)-\epsilon_{11}\Phi_1'(z)]+\mathrm{Im}\,[g_{11}\Omega'(z)]\end{cases}$$

$$\begin{cases}B_x = -\mathrm{Re}\,[\Omega'(z)], & H_x = \mathrm{Im}\,[f_{15}U_1'(z)-g_{11}\Phi_1'(z)]-\mathrm{Re}\,[\mu_{11}\Omega'(z)] \\ B_y = \mathrm{Im}\,[\Omega'(z)], & H_y = \mathrm{Re}\,[f_{15}U_1'(z)-g_{11}\Phi_1'(z)]+\mathrm{Im}\,[\mu_{11}\Omega'(z)].\end{cases} \tag{A6}$$

Solution 2

On account of the constitutive relation (5c), the transversely isotropic medium has

$$\{\boldsymbol{\sigma}\} = [\mathbf{C}^*]\{\boldsymbol{\varepsilon}\} - [\mathbf{e}^*]^{\mathrm{T}}\{\mathbf{E}\} + [\mathbf{f}^*]^{\mathrm{T}}\{\mathbf{H}\}$$
$$\{\mathbf{D}\} = [\mathbf{e}^*]\{\boldsymbol{\varepsilon}\} + [\boldsymbol{\epsilon}^*]\{\mathbf{E}\} + [\mathbf{g}^*]\{\mathbf{H}\}$$
$$\{\mathbf{B}\} = [\mathbf{f}^*]\{\boldsymbol{\varepsilon}\} - [\mathbf{g}^*]\{\mathbf{E}\} + [\boldsymbol{\mu}^*]\{\mathbf{H}\}, \tag{A7}$$

where

$$[\mathbf{C}^*] = \begin{bmatrix} c^*_{11} & c^*_{12} & c^*_{13} & 0 & 0 & 0 \\ c^*_{12} & c^*_{11} & c^*_{13} & 0 & 0 & 0 \\ c^*_{13} & c^*_{13} & c^*_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c^*_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c^*_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & (c^*_{11}-c^*_{12})/2 \end{bmatrix}$$

$$[\mathbf{e}^*] = \begin{bmatrix} 0 & 0 & 0 & 0 & e^*_{15} & 0 \\ 0 & 0 & 0 & e^*_{15} & 0 & 0 \\ e^*_{31} & e^*_{31} & e^*_{33} & 0 & 0 & 0 \end{bmatrix}, \quad [\mathbf{f}^*] = \begin{bmatrix} 0 & 0 & 0 & 0 & f^*_{15} & 0 \\ 0 & 0 & 0 & f^*_{15} & 0 & 0 \\ f^*_{31} & f^*_{31} & f^*_{33} & 0 & 0 & 0 \end{bmatrix}$$

$$[\boldsymbol{\epsilon}^*] = \begin{bmatrix} \epsilon^*_{11} & 0 & 0 \\ 0 & \epsilon^*_{11} & 0 \\ 0 & 0 & \epsilon^*_{33} \end{bmatrix}, \quad [\mathbf{g}^*] = \begin{bmatrix} g^*_{11} & 0 & 0 \\ 0 & g^*_{11} & 0 \\ 0 & 0 & g^*_{33} \end{bmatrix}$$

$$[\boldsymbol{\mu}^*] = \begin{bmatrix} \mu^*_{11} & 0 & 0 \\ 0 & \mu^*_{11} & 0 \\ 0 & 0 & \mu^*_{33} \end{bmatrix}.$$

If, assuming,

$$u_x = u_y = 0, \quad u_z = u_z(x,y) \quad \text{(out-of-plane deformation)}$$
$$\psi = \psi(x,y), \quad \varphi = \varphi(x,y) \tag{A8}$$

and substituting eqn (A7) into eqn (2d–f), we have

$$c^*_{44}\nabla^2 u_z + e^*_{15}\nabla^2\varphi - f^*_{15}\nabla^2\psi = 0$$
$$e^*_{15}\nabla^2 u_z - \epsilon^*_{11}\nabla^2\varphi - g^*_{11}\nabla^2\psi = 0$$
$$f^*_{15}\nabla^2 u_z + g^*_{11}\nabla^2\varphi - \mu^*_{11}\nabla^2\psi = 0. \tag{A9}$$

Due to the arbitariness of the coefficients, eqn (A9) leads to

$$\nabla^2 u_z = 0, \quad \nabla^2\varphi = 0, \quad \nabla^2\psi = 0. \tag{A10}$$

From eqn (A10), we have

$$u_z = \mathrm{Im}\,[U_2(z)], \quad \varphi = \mathrm{Im}\,[\Phi_2(z)], \quad \psi = \mathrm{Im}\,[\Psi(z)], \tag{A11}$$

where $U_2(z)$, $\Phi_2(z)$ and $\Psi(z)$ are complex potentials. It is worth mentioning that the assumption (A8) is not synonymous with eqn (A2). As a consequence, the strain, stress, electric field, electric displacement, magnetic induction and magnetic field are described by

$$\begin{cases} \gamma_{zx} = \mathrm{Im}\,[U'_2(z)], & \sigma_{zx} = \mathrm{Im}\,[c^*_{44}U'_2(z) + e^*_{15}\Phi'_2(z) - f^*_{15}\Psi'(z)] \\ \gamma_{zy} = \mathrm{Re}\,[U'_2(z)], & \sigma_{zy} = \mathrm{Re}\,[c^*_{44}U'_2(z) + e^*_{15}\Phi'_2(z) - f^*_{15}\Psi'(z)] \end{cases}$$
$$\begin{cases} E_x = -\mathrm{Im}\,[\Phi'_2(z)], & D_x = \mathrm{Im}\,[e^*_{15}U'_2(z) - \epsilon^*_{11}\Phi'_2(z) - g^*_{11}\Psi'(z)] \\ E_y = -\mathrm{Re}\,[\Phi'_2(z)], & D_y = \mathrm{Re}\,[e^*_{15}U'_2(z) - \epsilon^*_{11}\Phi'_2(z) - g^*_{11}\Psi'(z)] \end{cases}$$
$$\begin{cases} H_x = -\mathrm{Im}\,[\Psi'(z)], & B_x = \mathrm{Im}\,[f^*_{15}U'_2(z) + g^*_{11}\Phi'_2(z) - \mu^*_{11}\Psi'(z)] \\ H_y = -\mathrm{Re}\,[\Psi'(z)], & B_y = \mathrm{Re}\,[f^*_{15}U'_2(z) + g^*_{11}\Phi'^{2}(z) - \mu^*_{11}\Psi'(z)]. \end{cases} \tag{A12}$$

APPENDIX B: THE MODE III CRACK TIP FIELD

As shown in Fig. 1, if a crack in the infinite body suffers from out-of-plane deformation, two kinds of boundary conditions on its face are described by eqns (47) and (48). Clearly, the combination of eqns (A5) or (A11) with two kinds of boundary condition and application of the complex function theory will lead to the solution of the mode III crack tip field.

Solution 1

In light of the boundary condition (47) and eqn (A5), we can obtain the solution in the following form

$$U_1(z) = A_1(z^2-a^2)^{1/2}, \quad \Phi_1(z) = B_1(z^2-a^2)^{1/2}, \quad \Omega(z) = C_1(z^2-a^2)^{1/2}, \tag{B1}$$

where A, B and C are complex coefficients which are determined by a remote load. When $z \to a$, the crack tip field is given by

$$u_z = K^S\sqrt{\left(\frac{2r}{\pi}\right)}\sin\frac{\theta}{2},\quad \varphi = K^E\sqrt{\left(\frac{2r}{\pi}\right)}\sin\frac{\theta}{2},\quad A_z = K^B\sqrt{\left(\frac{2r}{\pi}\right)}\sin\frac{\theta}{2}$$

$$\gamma_{zx} = -\frac{K^S}{\sqrt{2\pi r}}\sin\frac{\theta}{2},\quad E_x = \frac{K^E}{\sqrt{2\pi r}}\sin\frac{\theta}{2},\quad B_x = -\frac{K^B}{\sqrt{2\pi r}}\cos\frac{\theta}{2}$$

$$\gamma_{zy} = \frac{K^S}{\sqrt{2\pi r}}\cos\frac{\theta}{2},\quad E_y = -\frac{K^E}{\sqrt{2\pi r}}\cos\frac{\theta}{2},\quad B_y = -\frac{K^B}{\sqrt{2\pi r}}\sin\frac{\theta}{2}$$

$$\sigma_{zx} = -\frac{c_{44}K^S+e_{15}K^E}{\sqrt{2\pi r}}\sin\frac{\theta}{2}+\frac{f_{15}K^B}{\sqrt{2\pi r}}\cos\frac{\theta}{2},\quad \sigma_{zy} = \frac{c_{44}K^S+e_{15}K^E}{\sqrt{2\pi r}}\cos\frac{\theta}{2}+\frac{f_{15}K^B}{\sqrt{2\pi r}}\sin\frac{\theta}{2}$$

$$D_x = -\frac{e_{15}K^S-\epsilon_{11}K^E}{\sqrt{2\pi r}}\sin\frac{\theta}{2}-\frac{g_{11}K^B}{\sqrt{2\pi r}}\cos\frac{\theta}{2},\quad D_y = \frac{e_{15}K^S-\epsilon_{11}K^E}{\sqrt{2\pi r}}\cos\frac{\theta}{2}-\frac{g_{11}K^B}{\sqrt{2\pi r}}\sin\frac{\theta}{2}$$

$$H_x = -\frac{f_{15}K^S-g_{11}K^E}{\sqrt{2\pi r}}\sin\frac{\theta}{2}-\frac{\mu_{11}K^B}{\sqrt{2\pi r}}\cos\frac{\theta}{2},\quad H_y = \frac{f_{15}K^S-g_{11}K^E}{\sqrt{2\pi r}}\cos\frac{\theta}{2}-\frac{\mu_{11}K^B}{\sqrt{2\pi r}}\sin\frac{\theta}{2}, \tag{B2}$$

where

$$K^S = A_1\sqrt{\pi a},\quad K^E = B_1\sqrt{\pi a},\quad K^B = C_1\sqrt{\pi a}.$$

If the boundary condition (48) is applied, it is found that $\Omega'(z)$ is an analytical function wherever z is, even on the crack, and

$$U_1(z) = A_2(z^2-a^2)^{-1/2},\quad \Phi_1(z) = B_2(z^2-a^2)^{-1/2},\quad \Omega(z) = C_2, \tag{B3}$$

where A_z is considered to be finite at $|z| \to \infty$. When $z \to a$, the crack tip field is given by

$$u_z = K^S\sqrt{\left(\frac{2r}{\pi}\right)}\sin\frac{\theta}{2},\quad \varphi = K^E\sqrt{\left(\frac{2r}{\pi}\right)}\sin\frac{\theta}{2},\quad A_z = \mathrm{Im}\,(c_2)$$

$$\gamma_{zx} = -\frac{K^S}{\sqrt{2\pi r}}\sin\frac{\theta}{2},\quad E_x = \frac{K^E}{\sqrt{2\pi r}}\sin\frac{\theta}{2},\quad B_x = 0$$

$$\gamma_{zy} = \frac{K^S}{\sqrt{2\pi r}}\cos\frac{\theta}{2},\quad E_y = -\frac{K^E}{\sqrt{2\pi r}}\cos\frac{\theta}{2},\quad B_y = 0$$

$$\sigma_{zx} = -\frac{c_{44}K^S+e_{15}K^E}{\sqrt{2\pi r}}\sin\frac{\theta}{2},\quad \sigma_{zy} = \frac{c_{44}K^S+e_{15}K^E}{\sqrt{2\pi r}}\cos\frac{\theta}{2}$$

$$D_x = -\frac{e_{15}K^S-\epsilon_{11}K^E}{\sqrt{2\pi r}}\sin\frac{\theta}{2},\quad D_y = \frac{e_{15}K^S-\epsilon_{11}K^E}{\sqrt{2\pi r}}\cos\frac{\theta}{2}$$

$$H_x = -\frac{f_{15}K^S-g_{11}K^E}{\sqrt{2\pi r}}\sin\frac{\theta}{2},\quad H_y = \frac{f_{15}K^S-g_{11}K^E}{\sqrt{2\pi r}}\cos\frac{\theta}{2}, \tag{B4}$$

where

$$K^S = A_2\sqrt{\pi a},\quad K^E = B_2\sqrt{\pi a}.$$

Solution 2

Combining eqn (A11) and the boundary condition (47), we find that $\Psi(z)$ is an analytical function everywhere, and

$$U_2(z) = A_3(z^2-a^2)^{-1/2},\quad \Phi_2(z) = B_3(z^2-a^2)^{-1/2},\quad \Psi(z) = C_3, \tag{B5}$$

where H_x is taken to be finite on $|z| \to \infty$. When $z \to a$, the crack tip field is

$$u_z = K^S\sqrt{\left(\frac{2r}{\pi}\right)}\sin\frac{\theta}{2},\quad \varphi = K^E\sqrt{\left(\frac{2r}{\pi}\right)}\sin\frac{\theta}{2},\quad \psi = \mathrm{Im}\,(c_3)$$

$$\gamma_{zx} = -\frac{K^S}{\sqrt{2\pi r}}\sin\frac{\theta}{2},\quad E_x = \frac{K^E}{\sqrt{2\pi r}}\sin\frac{\theta}{2},\quad H_x = 0$$

$$\gamma_{zy} = \frac{K^S}{\sqrt{2\pi r}}\cos\frac{\theta}{2},\quad E_y = -\frac{K^E}{\sqrt{2\pi r}}\cos\frac{\theta}{2},\quad H_y = 0$$

$$\sigma_{zx} = -\frac{c_{44}^{*}K^{S}+e_{15}^{*}K^{E}}{\sqrt{2\pi r}}\sin\frac{\theta}{2},\quad \sigma_{zy} = \frac{c_{44}^{*}K^{S}+e_{15}^{*}K^{E}}{\sqrt{2\pi r}}\cos\frac{\theta}{2}$$

$$D_{x} = -\frac{e_{15}^{*}K^{S}-\epsilon_{11}^{*}K^{E}}{\sqrt{2\pi r}}\sin\frac{\theta}{2},\quad D_{y} = \frac{e_{15}^{*}K^{S}-\epsilon_{11}^{*}K^{E}}{\sqrt{2\pi r}}\cos\frac{\theta}{2}$$

$$B_{x} = -\frac{f_{15}^{*}K^{S}+g_{11}^{*}K^{E}}{\sqrt{2\pi r}}\sin\frac{\theta}{2},\quad B_{y} = \frac{f_{15}^{*}K^{S}+g_{11}^{*}K^{E}}{\sqrt{2\pi r}}\cos\frac{\theta}{2}, \tag{B6}$$

where

$$K^{S} = A_{3}\sqrt{\pi a},\quad K^{E} = B_{3}\sqrt{\pi a}.$$

On account of the boundary condition (48) and eqn (A11), we can obtain the solution in the following form

$$U_{2}(z) = A_{4}(z^{2}-a^{2})^{1/2},\quad \Phi_{2}(z) = B_{4}(z^{2}-a^{2})^{1/2},\quad \Psi(z) = C_{4}(z^{2}-a^{2})^{1/2}. \tag{B7}$$

When $z \to a$, the crack tip field is given by

$$u_{z} = K^{S}\sqrt{\left(\frac{2r}{\pi}\right)}\sin\frac{\theta}{2},\quad \varphi = K^{E}\sqrt{\left(\frac{2r}{\pi}\right)}\sin\frac{\theta}{2},\quad \psi = K^{H}\sqrt{\left(\frac{2r}{\pi}\right)}\sin\frac{\theta}{2}$$

$$\gamma_{zx} = -\frac{K^{S}}{\sqrt{2\pi r}}\sin\frac{\theta}{2},\quad E_{x} = \frac{K^{E}}{\sqrt{2\pi r}}\sin\frac{\theta}{2},\quad H_{x} = \frac{K^{H}}{\sqrt{2\pi r}}\sin\frac{\theta}{2}$$

$$\gamma_{zy} = \frac{K^{S}}{\sqrt{2\pi r}}\cos\frac{\theta}{2},\quad E_{y} = -\frac{K^{E}}{\sqrt{2\pi r}}\cos\frac{\theta}{2},\quad H_{y} = -\frac{K^{H}}{\sqrt{2\pi r}}\cos\frac{\theta}{2}$$

$$\sigma_{zx} = -\frac{c_{44}^{*}K^{S}+e_{15}^{*}K^{E}-f_{15}^{*}K^{H}}{\sqrt{2\pi r}}\sin\frac{\theta}{2},\quad \sigma_{zy} = \frac{c_{44}^{*}K^{S}+e_{15}^{*}K^{E}-f_{15}^{*}K^{H}}{\sqrt{2\pi r}}\cos\frac{\theta}{2}$$

$$D_{x} = -\frac{e_{15}^{*}K^{S}-\epsilon_{11}^{*}K^{E}-g_{11}^{*}K^{H}}{\sqrt{2\pi r}}\sin\frac{\theta}{2},\quad D_{y} = \frac{e_{15}^{*}K^{S}-\epsilon_{11}^{*}K^{E}-g_{11}^{*}K^{H}}{\sqrt{2\pi r}}\cos\frac{\theta}{2}$$

$$B_{x} = -\frac{f_{15}^{*}K^{S}+g_{11}^{*}K^{E}-\mu_{11}^{*}K^{H}}{\sqrt{2\pi r}}\sin\frac{\theta}{2},\quad B_{y} = \frac{f_{15}^{*}K^{S}+g_{11}^{*}K^{E}-\mu_{11}^{*}K^{H}}{\sqrt{2\pi r}}\cos\frac{\theta}{2}, \tag{B8}$$

where

$$K^{S} = A_{4}\sqrt{\pi a},\quad K^{E} = B_{4}\sqrt{\pi a},\quad K^{H} = C_{4}\sqrt{\pi a}.$$